

Theoretical Study of Time Correlation Functions in a Discrete Chaotic Process

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One-dimensional discrete chaotic processes are studied from a statistical-dynamical point of view. A set of equations which describe the behavior of the time correlation functions is derived with the aid of Mori's projector formalism. A condition under which a process is Markoffian is obtained, and an approximate method is developed for a non-Markoffian process. As an illustration, a time correlation function for a simple system is calculated and the comparison with results of computer simulations is made. The relation between the instability of a trajectory and the characteristic time of chaotic motions is also discussed.

I. Introduction

Turbulence has been the subject of many theoretical and experimental studies [1–4]. The most important feature of recent theories is the concept of deterministic nonperiodic motions obeying ordinary differential equations [1–3]. This concept was first used by Lorenz [1] in his model for the Bénard-Rayleigh convection. A dynamical system which exhibits turbulence shows a strong instability of its trajectory in phase space: the separation distance between two initially-nearby trajectories grows exponentially in time, and the trajectory is extremely sensitive to the initial state. As a result, the time correlation vanishes as the time difference tends to infinity. Lorenz has also found that temporally-local maxima of one variable obey an approximate one-dimensional difference equation. This fact means that certain kinds of deterministic difference systems also show chaotic behaviors. Since Lorenz's work [1], discrete chaos has become of great interest in ecological and chemical contexts [5].

A one-dimensional discrete system is described by the difference equation

$$X_{t+1} = F(X_t), \quad (\text{I.1})$$

where t indicates a discrete time. It depends on the shape of $F(x)$ whether (I.1) has a chaotic solution or not [6]. Hereafter we assume that (I.1) has a chaotic solution. In the chaotic difference system,

strong instability of the trajectory* and disappearance of the time correlation are observed as in the differential case.

One of the main purposes of this paper is to construct a statistical-dynamical approach to the time correlation functions of chaotic motions of the difference system (I.1). In II. the statistical approach is described. A formulation is given in III. in order to study the time correlations. As illustrations, a few examples are considered in IV., and a summary and a discussion are given in V.

II. Statistical Description of Chaos

In the chaotic state, because of the strong instability of the trajectory, the power spectrum of the state variable becomes broad and many modes with various time scales are excited, which stiffly couple to each other. In this sense, the problem may be regarded as a many-body problem. Also due to the strong trajectory instability, the initial memory of the system is rapidly washed out when we observe time-averaged quantities. Therefore, the system should be treated statistically.

Let us start with (I.1), which leads to

$$\delta(X_{t+1} - x) = \mathcal{H} \delta(X_t - x), \quad (\text{II.1})$$

where \mathcal{H} is the Frobenius-Perron operator defined by

$$\begin{aligned} \mathcal{H} G(x) &\equiv \int \delta(F(y) - x) G(y) dy \\ &= \sum_j G(x_j) / |F'(x_j)| \end{aligned} \quad (\text{II.2})$$

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* The terminology "trajectory" means a sequence $\{X_0, X_1, X_2, \dots\}$ starting at a certain initial value X_0 .



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for an arbitrary function G . Here x_j is the j -th solution of $F(x_j) = x$ and $F'(x)$ denotes $dF(x)/dx$. Taking the time average of (II.1), one has the master equation

$$P_{t+1}(x) = \mathcal{H} P_t(x), \quad (\text{II.3})$$

where

$$P_t(x) \equiv \lim_{T \rightarrow \infty} (1/T) \sum_{s=1}^T \delta(X_{t+s} - x)$$

is the *probability distribution* that X_t has a value x at the time t . Hereafter we assume that, as t tends to infinity, $P_t(x)$ approaches a unique function $P^*(x)$. Thus the steady chaotic state is specified by the steady probability distribution $P^*(x)$, which is the solution of

$$P^*(x) = \mathcal{H} P^*(x). \quad (\text{II.4})$$

Chaotic motions have been studied with several quantities; e.g., the Kolmogorov entropy [7] and the rate of divergence of initially-nearby trajectories [8]. The Kolmogorov entropy plays a central role in mathematics and is known to be connected with the rate of divergence of initially-nearby trajectories which is sketched in Appendix. In the following section, the relaxation process of chaotic motions is investigated with the use of the projector method of Mori [9]. To characterize the chaos, the decay rate of the time correlation is introduced, and the connection between the divergence rate and the decay rate is also investigated. It should be noted that throughout the present paper irregular, chaotic motions will be referred to as “fluctuations” and are regarded as the intrinsic fluctuations in the steady state specified by $P^*(x)$.

III. Time Correlation Function in a Steady State

a) Equations of Motion in a Reduced Form

Let $A(x)$ and $B(x)$ be arbitrary functions of x and define the time correlation function by

$$\begin{aligned} G_t^{AB} &\equiv \overline{A(X_{t+s}) B(X_s)} \\ &\equiv \lim_{T \rightarrow \infty} (1/T) \sum_{s=1}^T A(X_{t+s}) B(X_s), \quad (\text{III.1}) \\ &= \langle (\mathcal{L}^t A(x)) B(x) \rangle, \end{aligned}$$

where $\langle \cdots \rangle$ means the average over $P^*(x)$, and \mathcal{L} is the time evolution operator introduced in the Appendix. Hereafter we consider the case $A = B = x - \langle x \rangle \equiv \delta x$.

A fluctuation defined by $\delta x_t \equiv x_t - \langle x \rangle$ ($x_t \equiv \mathcal{L}^t x$) obeys

$$\delta x_{t+1} = \mathcal{L} \delta x_t. \quad (\text{III.2})$$

By making use of the operator identity

$$\begin{aligned} \mathcal{L}^{t+1} &= \mathcal{L}^t \mathcal{P} \mathcal{L} + \sum_{s=0}^{t-1} \mathcal{L}^{t-1-s} \mathcal{P} \mathcal{L} (\mathcal{P}' \mathcal{L})^s \mathcal{P}' \mathcal{L} \\ &\quad + (\mathcal{P}' \mathcal{L})^t \mathcal{P}' \mathcal{L} \end{aligned} \quad (\text{III.3})$$

with the projectors \mathcal{P} and \mathcal{P}' defined by

$$\begin{aligned} \mathcal{P} G(x) &\equiv \delta x \langle G(x) \delta x \rangle / \langle (\delta x)^2 \rangle, \\ \mathcal{P}' &\equiv 1 - \mathcal{P}, \end{aligned} \quad (\text{III.4})$$

(III.2) is exactly transformed into

$$\delta x_{t+1} = \xi_1 \delta x_t + \sum_{s=0}^{t-1} \psi_{t-1-s} \delta x_s + f_t, \quad (\text{III.5})$$

where

$$\begin{aligned} \xi_1 &\equiv \langle \delta F(x) \delta x \rangle / \langle (\delta x)^2 \rangle, \\ \psi_t &\equiv \langle (\mathcal{L} f_t) \delta x \rangle / \langle (\delta x)^2 \rangle, \\ f_t &\equiv (\mathcal{P}' \mathcal{L})^t \mathcal{P}' \delta F(x). \end{aligned} \quad (\text{III.6})$$

Some remarks on (III.5) are to be noted: this equation has a structure similar to Mori's transport equation [9]. The first and second term on the r.h.s. correspond to a bare damping (frequency) and a renormalized (memory) term, respectively. The last term f_t represents the *fluctuating force* which maintains the steady chaotic state and is orthogonal to δx in the sense that $\langle f_t \delta x \rangle = 0$ ($t \geq 0$). The fluctuating force f_t vanishes if and only if $F(x)$ is a linear function of x . Hence the fluctuation force f_t always exists in the chaotic case. Defining the normalized time correlation (relaxation) function ξ_t^* by

$$\xi_t = \langle \delta x_t \delta x \rangle / \langle (\delta x)^2 \rangle, \quad \xi_0 = 1, \quad (\text{III.7})$$

one obtains from (III.5)

$$\xi_{t+1} = \xi_1 \xi_t + \sum_{s=0}^{t-1} \psi_{t-1-s} \xi_s^{**}. \quad (\text{III.8})$$

* From the relation $\langle (x_t - x)^2 \rangle = 2 \langle (\delta x)^2 \rangle (1 - \xi_t)$, ξ_t is always smaller than unity.

** By introducing a Laplace transform

$$\xi[z] \equiv \sum_{t=0}^{\infty} \xi_t \exp(-zt),$$

this can be solved to yield

$$\xi_t = \int_{-\pi}^{\pi} \xi[i\omega] \exp(i\omega t) d\omega / 2\pi,$$

where

$$\xi[z] = [1 - \xi_1 \exp(-z) - \psi[z] \exp(-2z)]^{-1}.$$

Furthermore, in terms of the quantity

$$\tilde{f} \equiv \mathcal{P}' P^{*-1} \mathcal{H}[P^*(x) \delta x], \quad (\text{III.9})$$

the memory function ψ_t is written as

$$\psi_t = \langle f_t \tilde{f} \rangle / \langle (\delta x)^2 \rangle. \quad (\text{III.10})$$

This expression for ψ_t is more tractable than the one in (III.6).

For certain systems $\tilde{f} = 0$. Then (III.5) reduces to

$$\delta x_{t+1} = \xi_1 \delta x_t + f_t, \quad (\text{III.11})$$

and the time correlation function vanishes according to

$$\xi_t = (\text{sgn}(\xi_1))^t \exp(-t/\tau_0),$$

where $\tau_0 (\equiv -1/\ln|\xi_1|)$ represents a decay time of fluctuations. The two terms on the r.h.s. of (III.11) play different roles: the first term reduces the fluctuation, while the second excites it. A *fluctuation-dissipation theorem* due to the balance between these two effects holds as follows:

$$\langle f_t f_{t'} \rangle / \langle (\delta x)^2 \rangle = (1 - \xi_1^2) \delta_{t,t'}, \quad (\text{III.12})$$

where $\delta_{t,t'}$ is the Kronecker delta. Thus f_t is exactly a white noise, but is generally *non-Gaussian*. Hereafter, the condition $\tilde{f} = 0$ is referred to as the *Markoff condition*.

On the other hand, for a system which does not satisfy the Markoff condition the memory effect may play a significant role. Roughly speaking, since ξ_t decays mainly due to the first term in (III.5), it is expected that for $|\xi_1| < 1/e$ the memory effect is small; while for $1/e < |\xi_1| < 1$ we must take into account this effect. When the memory effect is not negligible, the memory function may be expanded with the continued fraction method of Mori [10].

b) Continued Fraction Expansion and Approximations

We use the expression (III.10) for the memory function. By introducing a vector $\mathbf{f}_t \equiv \mathcal{L}_1^t \begin{pmatrix} f_0 \\ \tilde{f} \end{pmatrix}$ with $\mathcal{L}_1 \equiv \mathcal{P}' \mathcal{L}$, one has

$$\mathbf{f}_{t+1} = \mathcal{L}_1 \mathbf{f}_t. \quad (\text{III.13})$$

This can be rewritten as

$$\mathbf{f}_{t+1} = \hat{\eta}_1 \cdot \mathbf{f}_t + \sum_{s=0}^{t-1} \hat{\psi}_{t-1-s}^{(1)} \cdot \mathbf{f}_s + \mathbf{g}_t \quad (\text{III.14})$$

with

$$\begin{aligned} \hat{\eta}_1 &\equiv \langle \mathbf{f}_1 \mathbf{f}_0^+ \rangle \cdot \langle \mathbf{f}_0 \mathbf{f}_0^+ \rangle^{-1}, \\ \hat{\psi}_t^{(1)} &\equiv \langle (\mathcal{L}_1 \mathbf{g}_t) \mathbf{f}_0^+ \rangle \cdot \langle \mathbf{f}_0 \mathbf{f}_0^+ \rangle^{-1} \quad \text{and} \\ \mathbf{g}_t &\equiv (\mathcal{P}_1' \mathcal{L}_1)^t \mathcal{P}_1' \mathcal{L}_1 \mathbf{f}_0, \end{aligned}$$

where $\hat{\eta}_1$ and $\hat{\psi}_t^{(1)}$ are 2×2 matrices. Here the projectors \mathcal{P}_1 and \mathcal{P}_1' have been defined by

$$\mathcal{P}_1 \mathbf{G}(x) = \langle \mathbf{G}(x) \mathbf{f}_0^+ \rangle \cdot \langle \mathbf{f}_0 \mathbf{f}_0^+ \rangle^{-1} \cdot \mathbf{f}_0$$

and $\mathcal{P}_1' \equiv 1 - \mathcal{P}_1$ for an arbitrary vector \mathbf{G} , and $+$ indicates the transpose. ψ_t is now given by the elements of $\hat{\eta}_t$ defined by $\hat{\eta}_t \equiv \langle \mathbf{f}_t \mathbf{f}_0^+ \rangle \cdot \langle \mathbf{f}_0 \mathbf{f}_0^+ \rangle^{-1}$, which is the solution of the equation

$$\hat{\eta}_{t+1} = \hat{\eta}_1 \cdot \hat{\eta}_t + \sum_{s=0}^{t-1} \hat{\psi}_{t-1-s}^{(1)} \cdot \hat{\eta}_s.$$

Now the new memory function $\hat{\psi}_s^{(1)}$ appears. In the same manner as the above, we have the equation of motion for $\hat{\psi}_s^{(1)}$, in which an unknown memory function $\hat{\psi}_s^{(2)}$ appears. In this way, in order to obtain ψ_t , an infinite series of equations of motion must be solved. In the Laplace transform, this expansion gives a continued fraction representation of the time correlation function [10].

However, an exact analysis of this infinite series of expansions is not easy, and certain approximations will be indispensable. The lowest order approximation is to neglect ψ_t . This will be referred to as Appendix I. The next simpler one is to discard the contributions higher than the third step of expansion, which means $\hat{\psi}_s^{(1)} = 0$. This corresponds to the second order perturbation approximation with respect to the deviation from the simple exponential decay with a decay time τ_0 , and will be called Appendix II. After a slight manipulation, the memory function in App. II is obtained as

$$\psi_t = c_1 \eta_1^t + c_2 \eta_2^t \quad (\text{III.15})$$

with

$$c_1 = [\langle f_0 \tilde{f} \rangle (\eta_1^{(11)} - \eta_2) + \langle \tilde{f} \tilde{f} \rangle] / \{ \langle (\delta x)^2 \rangle (\eta_1 - \eta_2) \}$$

and

$$c_2 = -[\langle f_0 \tilde{f} \rangle (\eta_1^{(11)} - \eta_1) + \langle \tilde{f} \tilde{f} \rangle] / \{ \langle (\delta x)^2 \rangle (\eta_1 - \eta_2) \},$$

where $\eta^{(ij)}$ and η_l ($l = 1, 2$) are the ij -element of $\hat{\eta}_1$ and its eigenvalues, respectively. If one of $|\eta_1|$ and $|\eta_2|$ is larger than unity, this approximation breaks down, and higher order contributions must be taken into account. We can estimate the strength of the memory effect with the quantity

$$v \equiv \text{Max}(|\langle f_0 \tilde{f} \rangle|, |\langle \tilde{f} \tilde{f} \rangle|) / \langle (\delta x)^2 \rangle.$$

For $v < 1$ the perturbation expansion may be correct. However, if $v > 1$, this approximation probably breaks down.

In order to characterize the time correlation function, it is convenient to introduce an effective decay time τ_{eff} by

$$\tau_{\text{eff}} = -1/\ln(1 - 1/A) \quad (\text{III.16})$$

with $A = \sum_{t=0}^{\infty} |\xi_t|$. For a Markoffian system, τ_{eff} is identical to τ_0 . It is expected that as the trajectory instability becomes strong (k increases), the decay time becomes short. Thus k and $1/\tau_{\text{eff}}$ (the decay rate of fluctuations) behave qualitatively in a similar way. In the following section, this qualitative relation between k and $1/\tau_{\text{eff}}$ will be examined with a simple model.

IV. Applications

First we consider a system described by $F(x) = x/\beta$ for $0 \leq x \leq \beta$ and $F(x) = (1-x)/(1-\beta)$ for $\beta < x \leq 1$. If also $P^*(x) = 1$ for $0 < x < 1$, then we have $k = -\beta \ln \beta - (1-\beta) \ln(1-\beta)$. For this system the Markoff condition exactly holds and the relaxation process can be described by (III.11) with $\xi_1 = 2\beta - 1$. This is a basic dynamics in studying discrete chaotic systems. The Markoff condition also holds for systems which are obtained by conjugation to this system [11]: e.g., a parabolic system $F(x) = 4x(1-x)$ is also Markoffian.

General systems, however, do not obey a Markoff process. In order to investigate the non-Markoff effect, let us consider a system $F(x) = \alpha x/\beta$ for $0 \leq x \leq \beta$ and $F(x) = \alpha(1-x)/(1-\beta)$ for $\beta < x \leq 1$ with $0 < \alpha \leq 1$. If $\alpha > \beta$ and $\alpha + \beta > 1$, the divergence rate becomes positive and the system exhibits a chaotic behavior. Furthermore, if

$$\beta = \{1 - \alpha + \sqrt{(1-\alpha)(1+3\alpha)}\}/2 \quad (\alpha > 2/3),$$

then the steady probability distribution is exactly calculated to yield $P^*(x) = (1-\beta)/\alpha(3\alpha-2)$ for $\alpha(1-\alpha)/(1-\beta) \leq x < \beta$ and $1/(3\alpha-2)$ for $\beta \leq x \leq \alpha$ **, the validity of which was examined by simulations. Hereafter we restrict ourselves to this particular case.

* This is equivalent to the condition under which the system has a period three solution β, α and $\alpha(1-\alpha)/(1-\beta)$ in this order.

** This probability distribution was first derived by Y. Oono.

In this system, it can be shown that the expansion parameter v and $|\eta_1|$ and $|\eta_2|$ are always smaller than unity, and the perturbation approximation discussed in III. is valid. The comparison between the theoretical and the numerically-obtained time correlation function*** for $\alpha = 0.8$ is displayed in Figure 1. The App. II gives an excellent agreement with the numerical result as compared with Appendix I. Also, the memory effect has a tendency to shorten the decay time. For another α , the qualitative features of the time correlation functions are the same as those in Fig. 1, and we conclude that the time correlation in this system decays almost exponentially rather than in a power form.

For the first example, we compared the divergence rate k with the decay time τ_0 and found that as k increases (decreases) due to the variation of β ,

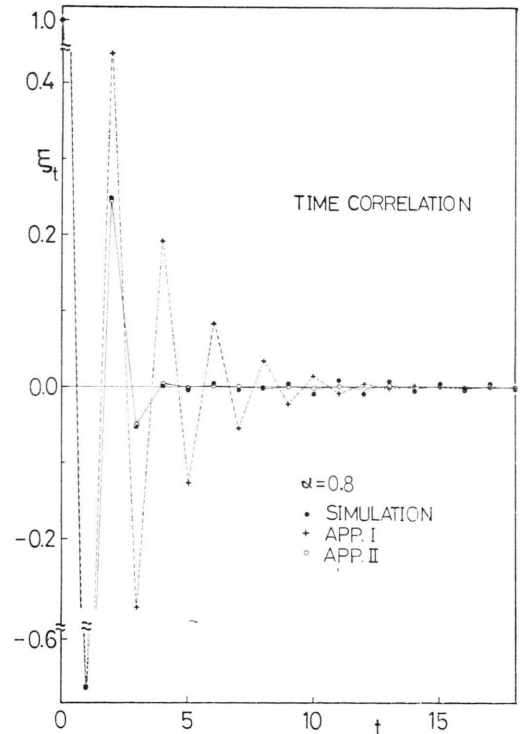


Fig. 1. Time correlation function for $\alpha = 0.8$. In this case, we have $v \cong 0.193$ and $|\eta_1| = |\eta_2| \cong 0.199$.

*** Computationally the time correlation function is obtained from the inverse Fourier transform of the power spectrum after 100 times of averaging. Here the power spectrum φ_ω was calculated by

$$\varphi_\omega = \sum_{t=-\infty}^{\infty} \overline{\delta X_{t+s} \delta X_s} \exp(-i\omega t),$$

where the bar indicates the time average (III.1).

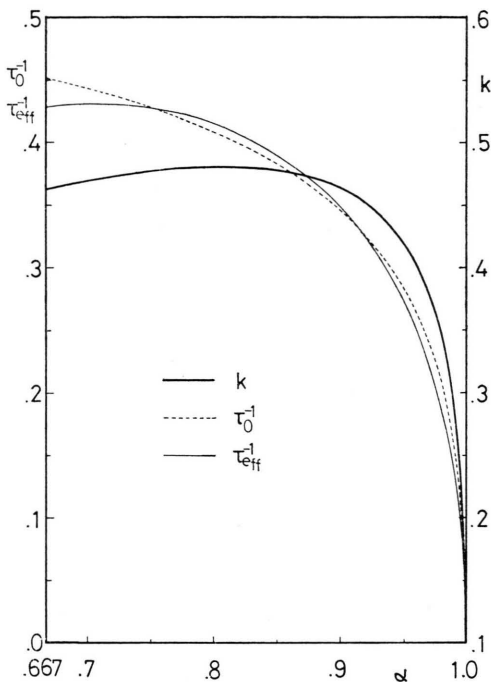


Fig. 2. Qualitative relations among the divergence rate k , the bare and effective decay rate, τ_0^{-1} and τ_{eff}^{-1} . k is calculated with the formula (A.2).

τ_0 becomes short (long). This relation between k and τ_0 coincides with the qualitative property mentioned in III. For the second example, k has a maximum near $\alpha = 0.81$. If we neglect the memory effect, the decay rate is given by $\tau_0^{-1} (\equiv -\ln |\xi_1|)$. The behavior of τ_0^{-1} displayed in Fig. 2 does not agree with the discussion in the previous section. However, this can be dissolved with the use of τ_{eff} introduced in (III.16). The effective decay rate τ_{eff}^{-1} derived by making use of the App. II is shown in Figure 2. Note that near $\alpha = 0.7$, τ_{eff}^{-1} has a maximum and the qualitative behavior of the decay rate tends to coincide with that of the divergence rate.

V. Summary and Discussion

In certain cases the Poincaré mapping of a differential chaos forms a systematic curve which corresponds to a one-dimensional discrete equation. The one-dimensional dynamics preserves the essential properties of the original one: e.g., the strength of the chaos and the behavior of the time correlation. Thus it is meaningful to study a relaxation process

of fluctuations in a one-dimensional chaos. In the present paper, the discrete chaos was investigated from the statistical-dynamical viewpoint.

The main results of the present paper are summarized as follows:

- 1) The equation which determines the behavior of the time correlation function was derived by using the projector method.
- 2) We obtained the condition under which the fluctuation decays exponentially, and an approximate method was given to study the memory effect.
- 3) For a simple system, the theoretical result for the time correlation function excellently agrees with that of a computer simulation, and it was found that the time correlation vanishes almost exponentially.
- 4) It seems that even for a non-Markoffian process a plausible relation between the divergence rate and the effective decay time exists: as the divergence rate increases (decreases), the decay time becomes short (long).

The result 4) implies that if r is an external parameter which controls the strength of a chaos, then the following relation may exist;

$$d\tau_{\text{eff}}(r)/dk(r) < 0. \quad (\text{V.1})$$

Here $k(r)$ and $\tau_{\text{eff}}(r)$ are the divergence rate and the decay time for the external parameter r . It can be shown that the first example in the previous section satisfies this relation. We believe that (V.1) holds for any other one-dimensional discrete chaos.

From a practical point of view, it is hard to solve the steady state equation (II.4) for an arbitrary function $F(x)$ which shows chaos. Therefore it is hoped to develop an approximate method that gives qualitatively correct results for the time correlation function without referring to the details of the steady distribution. In connection with this, it is also hoped to develop the approach for a multi-dimensional discrete chaos.

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Appendix

When a system has a chaotic solution, a trajectory becomes unstable. This is characterized by the rate of divergence of initially-nearby trajectories k .

The distance between two points started initially at X_0 and $X_0 + d_0$ is given by

$$|d_t| = |d_0| \int (dx_t/dx) \delta(X_0 - x) dx + O(|d_0|^2) \quad (\text{A.1})$$

with $x_t \equiv \mathcal{L}^t x$, where \mathcal{L} is a time evolution operator defined by $\mathcal{L}G(x) = G(F(x))$ for an arbitrary function $G(x)$. The rate of divergence of initially-nearby trajectories is defined by

$$k \equiv \langle \ln(|d_t|/|d_0|) \rangle_{X_0} / t$$

in the limit $|d_0| \rightarrow 0$, where $\langle \cdots \rangle_{X_0}$ is an average over the initial ensemble of X_0 . Suppose that the system is initially in the steady state, one has $\langle \delta(X_0 - x) \rangle_{X_0} = P^*(x)$. Thus we have $k = \langle K_t(x) \rangle$ with

$$K_t(x) \equiv \ln |dx_t/dx| / t,$$

where $\langle \cdots \rangle = \int dx P^*(x) \cdots$. Then the recursion relation for $K_t(x)$ is obtained as

$$(t+1) K_{t+1}(x) = \ln |F'(x)| + \ln |\mathcal{L} \exp \{t K_t(x)\}|.$$

Taking the average of both hand sides, one has

$$(t+1) \langle K_{t+1}(x) \rangle = \langle \ln |F'(x)| \rangle + t \langle K_t(x) \rangle.$$

Integrating this yields

$$k = \langle \ln |F'(x)| \rangle, \quad (\text{A.2})$$

which does not depend on t . Roughly speaking, the distance between two nearby points varies with time like $\propto \exp(kt)$. Thus the statement that the difference system (I.1) shows a chaos is physically equivalent to the existence of a positive k . The strength of the chaos becomes larger as k increases.

It should be noted that the above discussion holds for $|d_t| \ll 1$ and the exponential growth is observed for $t \ll t_0$, where $t_0 = \ln(1/|d_0|)/k$. After this period the separation distance is bounded due to the finiteness of the phase space.

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